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Higher loop Bethe ansatz for open spin-chains in AdS/CFT

Keisuke Okamura[a] **and Kentaroh Yoshida**[b]

[a] *Department of Physics, Faculty of Science, University of Tokyo Bunkyo-ku, Tokyo 113-0033, Japan*
[b] *Theory Division, Institute of Particle and Nuclear Studies High Energy Accelerator Research Organization (KEK) Tsukuba, Ibaraki 305-0801, Japan*
E-mail: okamura@hep-th.phys.s.u-tokyo.ac.jp, kyoshida@post.kek.jp

Abstract: We propose a perturbative asymptotic Bethe ansatz (PABA) for open spin-chain systems whose Hamiltonians are given by matrices of anomalous dimension for composite operators, and apply it to two types of composite operators related to two different brane configurations. One is an $\mathrm{AdS}_4 \times \mathrm{S}^2$-brane in the bulk $\mathrm{AdS}_5 \times \mathrm{S}^5$ which gives rise to a defect conformal field theory (dCFT) in the dual field theory, and the other is a giant graviton system with an open string excitation. In both cases, excitations on open strings attaching to D-branes (a D5-brane for the dCFT case, and a spherical D3-brane for the giant graviton case) can be represented by magnon states in the spin-chains with appropriate boundary conditions, in which informations of the D-branes are encoded. We concentrate on single-magnon problems, and explain how to calculate boundary S-matrices via the PABA technique. We also discuss the energy spectrum in the BMN limit.

Keywords: Penrose limit and pp-wave background, Bethe Ansatz, AdS-CFT Correspondence, D-branes.

Contents

1. Introduction

The AdS/CFT correspondence [1] has continued to give strong impacts on fundamental aspects of string theories and many applications to study non-perturbative aspects of gauge theories. It is, up to now, just a conjecture without rigorous proof, but there are many evidences to support it without obvious failure. Hence it is surely an important task to seek more evidences in some tractable laboratories.

Recently remarkable developments have been achieved in integrable structures on both sides of the duality. On the gauge theory side, Minahan and Zarembo [2] showed that a matrix of anomalous dimension of one-loop planar dilatation operator in the SO(6) sector is represented by a Hamiltonian of an integrable spin chain. From their pioneering work, much attention has been payed to integrable structures in the AdS/CFT correspondence. The classical integrability of a single system of a superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ plays an important role, and it is considered to be encoded into the integrable structure of the $\mathrm{PSU}(2,2|4)$ super spin-chain in the dual large N gauge theory [3]. Then the integrable structure is very helpful to study anomalous dimensions at higher order in perturbation theory, since one can use a so-called Bethe ansatz technique in field of integrable systems.

The Bethe ansatz technique has been developed and utilized in studies of large N gauge theories very efficiently [2, 4 − 19] (For an exhaustive review, see [20]). Based on these successful examples in closed-strings/closed spin-chains cases, much efforts have been made to push the ideas toward open-strings/open-chains cases [21 − 28], and striking matchings are shown at the one-loop level in an effective coupling $\lambda' \equiv g_{\mathrm{YM}}^2 N/L^2$. Here L is a large quantum number representing the total spin of a string on one side, and the R-charge of an operator super Yang-Mills (SYM) theory on the other. Such open string sectors are realized by inserting some D-branes (and orientifolds) into the bulk $\mathrm{AdS}_5 \times \mathrm{S}^5$, and there are many literatures dealing with the open string issues [21 − 32]. There a spin-chain has no longer a periodic boundary condition of a closed spin-chain case; instead of the periodicity, it has boundaries imposed by the D-brane setup. It is then not clear whether the integrability still holds or not with the boundaries. In some particular cases, the boundaries were shown to be integrable at least at the one-loop level [21, 23, 24, 26, 27], i.e., the system can be solved by the Bethe ansatz technique. When we consider higher-loop contributions to the boundaries, however, the problem gets more complicated and whether the integrable structures still remains or not is unclear. One of the aims of this paper is to examine such an integrability issue.

In comparison to the closed spin-chain case, in the open spin-chain case, boundary S-matrices appear in Bethe equations as new ingredients. When the magnon is scattered at either of the left/right boundaries, the Bethe wavefunction potentially gets a phase shift depending on the quasi-momentum of the magnon in general. Besides that, the wavefunction gets phase factors in colliding against other magnons in the bulk of the chain just as in the closed spin-chain case. The integrability assures that (i) the magnitude of the momentum does not change, either in interacting with another magnon, or in reflecting off the boundary, and (ii) the scattering of spin-waves factorizes into products of two-body scatterings. Let the number of the magnons in the open spin-chain be M, and the number of the sites in the chain L. Then the Bethe equation for the open spin-chain typically takes the form

$$e^{2ip_jL} = B_1(p_j;g)B_L(-p_j;g)\prod_{k\neq j,\ k=1}^{M}\left(S_{kj}\left(p_k,p_j;g\right)S_{kj}\left(p_k,-p_j;g\right)\right) \tag{1.1}$$

for $k = 1,\dots,M$, where S_{kj} is a two-body bulk S-matrix and $B_{1,L}$ are boundary S-matrices defined at $x = 1$ and L, respectively. When we consider higher-loop corrections, both kinds of matrices may depend on the gauge coupling g (defined in (2.1)), which we made explicit in (1.1). Toward higher loop analysis, the Bethe ansatz technique has to be generalized so that the S-matrices admit a g-expansion. Such a generalized Bethe ansatz for the closed spin-chain case is often referred to as *perturbative asymptotic Bethe ansatz (PABA)*. The PABA technique for the closed spin-chain was established by Staudacher [33] to derive the bulk S-matrix for SU(1|1) sector, and has been utilized in analyzing the integrable structures of SU(1|2) and SU(1,1|2) sectors [16], plane-wave matrix model [34, 35]. Recently it was also applied to a giant graviton system with an open string excitation by Agarwal [28]. There the PABA was generalized so that it was applicable to an open spin-chain with boundaries, and the potential breakdown of integrability at the two-loop order was suggested.

For the purpose of studying higher-loop contributions to open spin-chains, our aim here is to examine the integrability of two distinct open spin-chain models derived from two types of composite operators related to two different brane setups; one is a giant graviton system with an open string excitation, which is the same system as studied in [28], and the other is a defect conformal field theory (dCFT) dual to D3-D5 system. In both models, a bulk two-body S-matrix is given by the same one as in the four-dimensional $\mathcal{N} = 4$ SYM, and the only object to which we should pay a special attention is the boundary S-matrix. By analyzing the structure of the boundary S-matrix, which reflects the field theory structure in concern, we would be able to draw some valuable information on the string theory side in view of the AdS/CFT dictionary.

In order to determine the boundary S-matrices, all we have to consider is a single-magnon problem, since the scattering at boundaries cannot occur with more than one magnon. For this reason, in this paper, we focus upon a single-magnon problem, and leave the problems of potential difficulty in dealing with more than two magnons at higher loops as a future work.[1] In a single-trace operator case, inserting only one magnon in the trace leads to a trivial problem due to the constraint that the magnon momentum along the spin-chain must vanish, but in the present case of "open" operators, it is not trivial any more due to the presence of boundaries, which indeed makes the problem essentially a three-body problem, as noticed in [28].

This paper is organized as follows: In section 2, we propose a PABA for an open spin-chain whose "bulk" structure is determined from the structure of perturbation theory of $\mathcal{N}=4$ SYM. Then in section 3 and section 4, we will apply it to two particular models: an open spin-chain on a giant graviton, and an open spin-chain in dCFT. Section 5 is devoted to a summary and outlooks.

2. Perturbative Asymptotic Bethe Ansatz for Open Spin-Chains

In this section, we propose a PABA method for open spin-chains whose "bulk" part of the Hamiltonian consists of two complex holomorphic fields in the $\mathcal{N}=4$ SYM theory, i.e., the fields belonging to an SU(2) sector.

2.1 Open Spin-Chain Hamiltonian from 'Open' Diagrams in SYM Theory

The dilatation operator for single-trace operators in an SU(2) sector of $\mathcal{N}=4$ SYM can be represented by a Hamiltonian of an integrable closed spin-chain, and is given by, up to the third order [36, 37],

$$H_{\text{bulk}} = \sum_{r=1}^{\infty} g^r \mathcal{H}_r \,, \qquad g \equiv \frac{\lambda}{16\pi^2} \tag{2.1}$$

where $\lambda \equiv g_{\text{YM}}^2 N$ is the 't Hooft coupling. The first few Hamiltonian densities $\mathcal{H}_r$ are given by

$$\begin{aligned}
\mathcal{H}_1 &= 2\sum_{l=1}^{L-1} Q_{l,l+1} \,, \qquad \mathcal{H}_2 = -8\sum_{l=1}^{L-1} Q_{l,l+1} + 2\sum_{l=1}^{L-2} Q_{l,l+2} \,, \\
\mathcal{H}_3 &= 60\sum_{l=1}^{L-1} Q_{l,l+1} - 24\sum_{l=1}^{L-2} Q_{l,l+2} + 4\sum_{l=1}^{L-3} Q_{l,l+3} \\
&\quad + b_1\sum_{l=1}^{L-3} Q_{l,l+2}Q_{l+1,l+3} + b_2\sum_{l=1}^{L-3} Q_{l,l+3}Q_{l+1,l+2} + b_3\sum_{l=1}^{L-3} Q_{l,l+1}Q_{l+2,l+3} \,.
\end{aligned} \tag{2.2}$$

[1] A study in this direction has been done for the case of a giant graviton system in [28], and it would be interesting to further examine the problem.

Here we have defined $Q_{l,k} \equiv \frac{1}{2}(1 - \vec{\sigma}_l \cdot \vec{\sigma}_k)$ with Pauli matrices $\{\vec{\sigma}_k\}_{k=1,2,3}$ for convenience. The three unknown coefficients $b_{1,2,3}$, which are sensitive to magnon interactions, cannot be determined by just demanding the BMN scaling[2]. As mentioned in the introduction, we concentrate on a single-magnon problem, so the QQ-terms in (2.2) are irrelevant to our present analysis. Hence, by omitting them and rearranging other Q-terms with respect to k in $Q_{l,l+k}$, the effective "bulk" Hamiltonian for the open spin-chain can be cast into the following form,

$$H'_{\text{bulk}} = \sum_{l=1}^{L-1} c_1(g) Q_{l,l+1} + \sum_{l=1}^{L-2} c_2(g) Q_{l,l+2} + \sum_{l=1}^{L-3} c_3(g) Q_{l,l+3} + \dots , \tag{2.3}$$

where the coupling-dependent coefficients $c_k(g)$ are given by

$$c_1(g) = 2g - 8g^2 + 60g^3 + \dots , \tag{2.4}$$

$$c_2(g) = 2g^2 - 24g^3 + \dots , \tag{2.5}$$

$$c_3(g) = 4g^3 + \cdots . \tag{2.6}$$

On the other hand, the "boundary" Hamiltonian depends on particular field theory structures which are determined by brane setups concerned. We will later see concrete two examples — one is derived from the giant graviton operators dual to an open string attaching to a giant graviton (section 3) and the other from the defect operator dual to an open string attaching to an AdS-brane (D5-brane) in the bulk [39] (section 4). In both cases, we will consider an SU(2) sector where the bulk part of the open spin-chains consists of only two of three kinds of holomorphic scalars in $\mathcal{N} = 4$ SYM (In the dCFT case, in addition to the bulk four-dimensional fields, three-dimensional fundamental scalars are attached to each end of the composite operators).

Let the number of the sites in the chain, or the length of the chain, L, and label the sites as $x = 1, 2, \dots, L$. As far as we consider only one magnon in the chain, the possible structures of Feynman diagrams can be classified by specifying where the magnon starts and where it ends through the boundary interactions. The notation $C_{x,x+s}(g)$ is convenient to indicate the diagrams in which the magnon starts at site x and end at $x + s$ (see figure 1). The change of the location of the magnon, s, can either be positive or negative, but $1 \le x + s \le L$ must be satisfied for the magnon not to go out of the spin-chain. The boundary Hamiltonian is obtained by summing over all possible diagrams,

$$H_{\text{boundary}} = \sum C_{x,x+s}(g) \, . \tag{2.7}$$

Note that, because of the Hermiticity of the Hamiltonian, if $C_{i,j}$ is contained in (2.7), so is $C_{j,i}$, and these two give rise to the same contributions.

Now we should ask: what diagrams can contribute to $C_{x,x+s}(g)$ up to some fixed order in g? The following two points should be taken into account. (i) The higher order in

[2] If we assume the integrability in addition to the BMN scaling, however, the coefficients can be fixed as $b_1 = 4$, $b_2 = -4$ and $b_3 = 0$ [36]. This prediction has recently been indeed confirmed by a direct perturbative calculation [38].

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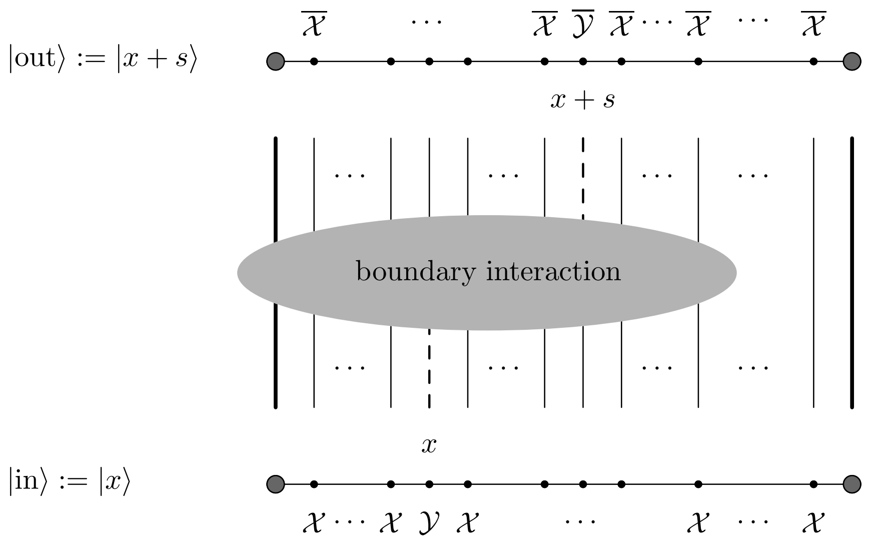

Figure 1: The diagram for $C_{x,x+s}(g)$. The shaded area represents the boundary interactions, through which the location of the impurity changes from $l = x$ to $l = x + s$ (the magnon state is represented by a broken line). This contribution leads to the dilatation operator for "open" composite SYM operators with one impurity $\mathcal{Y}$. Two gray dots on each ends of the chain stand for the boundary fields.

perturbation we proceed, the wider region around each boundary the magnon would feel as a scattering area. (ii) If the magnon in its in-state locates at $l = x$, only more than $|s|$-loop effect can take it to $l = x + s$ in its out-state. With these in mind, we expect the boundary Hamiltonian that is relevant to our single-magnon problem to be given by

$$\begin{aligned}
(2.7) &= \sum_k g^k C^{(k)}_{x,x+s} \qquad (2.8)\\
&= g\left(C^{(1)}_{1,1} + C^{(1)}_{L,L}\right) + g^2\left(C^{(2)}_{1,1} + C^{(2)}_{1,2} + C^{(2)}_{2,1} + C^{(2)}_{2,2} + C^{(2)}_{L,L} + C^{(2)}_{L,L-1} + C^{(2)}_{L-1,L} + C^{(2)}_{L-1,L-1}\right) +\\
&\quad + g^3\Big(C^{(3)}_{1,1} + C^{(3)}_{1,2} + C^{(3)}_{1,3} + C^{(3)}_{2,1} + C^{(3)}_{2,2} + C^{(3)}_{2,3} + C^{(3)}_{3,1} + C^{(3)}_{3,2} + C^{(3)}_{3,3} + C^{(3)}_{L,L} + C^{(3)}_{L,L-1} +\\
&\quad + C^{(3)}_{L,L-2} + C^{(3)}_{L-1,L} + C^{(3)}_{L-1,L-1} + C^{(3)}_{L-1,L-2} + C^{(3)}_{L-2,L} + C^{(3)}_{L-2,L-1} + C^{(3)}_{L-2,L-2}\Big) + \mathcal{O}(g^4)\,.
\end{aligned}$$

As in (2.8), each $C_{x,x+s}(g)$ will be perturbatively expanded in g. Our strategy here is to investigate the structure of boundary S-matrices at higher loops without fixing the coefficients $C^{(k)}_{x,x+s}$, so that it can be used to discuss various open spin-chain models. The total Hamiltonian of the open spin-chain is then given by the sum of the bulk (2.3) and the boundary part (2.7),

$$H \equiv H'_{\mathrm{bulk}} + H_{\mathrm{boundary}}\,. \qquad (2.9)$$

Up to the third order, by acting the bulk Hamiltonian (2.3) and the boundary Hamiltonian (2.7) to single-magnon sates $|x\rangle$ $(x = 1, 2, 3, \ldots)$, we obtain

$$
\begin{aligned}
H\,|1\rangle &= c_1(g)\,(|1\rangle - |2\rangle) + c_2(g)\,(|1\rangle - |3\rangle) + c_3(g)\,(|1\rangle - |4\rangle) + \\
&\quad + \left(gC^{(1)}_{1,1} + g^2C^{(2)}_{1,1} + g^3C^{(3)}_{1,1}\right)|1\rangle + \left(g^2C^{(2)}_{1,2} + g^3C^{(3)}_{1,2}\right)|2\rangle + g^3C^{(3)}_{1,3}\,|3\rangle + \mathcal{O}(g^4)\,, \qquad (2.10)\\
H\,|2\rangle &= c_1(g)\,(-\,|1\rangle + 2\,|2\rangle - |3\rangle) + c_2(g)\,(|2\rangle - |4\rangle) + c_3(g)\,(|2\rangle - |5\rangle) + \\
&\quad + \left(g^2C^{(2)}_{2,1} + g^3C^{(3)}_{2,1}\right)|1\rangle + \left(g^2C^{(2)}_{2,2} + g^3C^{(3)}_{2,2}\right)|2\rangle + g^3C^{(3)}_{2,3}\,|3\rangle + \mathcal{O}(g^4)\,, \qquad (2.11)\\
H\,|3\rangle &= c_1(g)\,(-\,|2\rangle + 2\,|3\rangle - |4\rangle) + c_2(g)\,(-\,|1\rangle + 2\,|3\rangle - |5\rangle) + c_3(g)\,(|3\rangle - |6\rangle) + \\
&\quad + g^3C^{(3)}_{3,1}\,|1\rangle + g^3C^{(3)}_{3,2}\,|2\rangle + g^3C^{(3)}_{3,3}\,|3\rangle + \mathcal{O}(g^4)\,, \qquad (2.12)\\
H\,|4\rangle &= c_1(g)\,(-\,|3\rangle + 2\,|4\rangle - |5\rangle) + c_2(g)\,(-\,|2\rangle + 2\,|4\rangle - |6\rangle) + c_3(g)\,(-\,|1\rangle + 2\,|4\rangle - |7\rangle) + \\
&\quad + \mathcal{O}(g^4)\,, \qquad (2.13)
\end{aligned}
$$

and so on. In the above expressions, the coefficients $c_k(g)$ contain the terms up to $\mathcal{O}(g^3)$; explicitly,

$$
c_1(g) = gc^{(1)}_1 + g^2c^{(2)}_1 + g^3c^{(3)}_1\,, \qquad c_2(g) = g^2c^{(2)}_2 + g^3c^{(3)}_2\,, \qquad c_3(g) = g^3c^{(3)}_3\,. \qquad (2.14)
$$

The dispersion relation can be determined only through bulk information, and is given by, up to n-loop order,

$$
\begin{aligned}
E(p,g) &= c_1(g)\left(2 - e^{ip} - e^{-ip}\right) + c_2(g)\left(2 - e^{2ip} - e^{-2ip}\right) + \cdots \\
&\qquad \cdots + c_n(g)\left(2 - e^{nip} - e^{-nip}\right) + \mathcal{O}(g^{n+1}) \\
&\equiv \sum_{k=1}^{n} \epsilon_k(p) g^k + \mathcal{O}(g^{n+1})\,. \qquad (2.15)
\end{aligned}
$$

Note that we have not yet expanded p in g for the time being, though it may depend on g. With our Hamiltonian (2.9), the first few energy coefficients turn out to be,

$$
\epsilon_1(p) = 8\sin^2\left(\frac{p}{2}\right), \quad \epsilon_2(p) = -32\sin^4\left(\frac{p}{2}\right), \quad \ldots\ . \qquad (2.16)
$$

Our next task is to solve an eigenvalue problem, and it will be discussed in the next subsection.

2.2 Bethe Wavefunction and Boundary S-matrix

In the SU(2) sector the "bulk" part of the open spin-chain consists of two kinds of complex holomorphic scalars. Let us denote them as $\mathcal{X}$ and $\mathcal{Y}$, and the letter is regarded as the magnon field. For the case of an SU(2) open spin-chain derived from a giant graviton operators (section 3), $\mathcal{Y}$ will be identified with the scalar field that form a determinant-like operator (i.e., a giant graviton on the gravity side).[3] For the case of an SU(2) open spin-chain derived from a defect operator (section 4), on the other hand, $\mathcal{Y}$ will be a holomorphic complex scalar field in the SO(3) vectormultiplet.

[3] This is not the only way to take an SU(2) sector; there is another possibility, in which the magnon state corresponds to an excitation in a Neumann direction to the giant graviton.

For a magnon at site x, the eigenvalue equation becomes

$$H|\Psi\rangle = E|\Psi\rangle \tag{2.17}$$

with the Hamiltonian (2.9) and the trial eigenfunction

$$|\Psi(p)\rangle = \sum_{1\le x\le L} \psi(x)\ |\mathcal{X}\ldots\mathcal{X}\overset{\overset{x}{\downarrow}}{\mathcal{Y}}\mathcal{X}\ldots\mathcal{X}\rangle \equiv \sum_{1\le x\le L} \psi(x)\ |x\rangle\ , \tag{2.18}$$

where we have defined the state with one impurity, or a magnon, $\mathcal{Y}$ at x as $|x\rangle$. We propose the following ansatz for the Bethe wavefunction on the open spin-chain:

$$\begin{aligned}\psi(x) = \Big(1 + f\left(|x-1|\,,p;g\right) + f\left(|L-x|\,,p;g\right)\Big)A(p;g)\,e^{ipx} - \\ - \Big(1 + \widetilde{f}\left(|x-1|\,,-p;g\right) + \widetilde{f}\left(|L-x|\,,-p;g\right)\Big)\widetilde{A}(-p;g)\,e^{-ipx}\,.\end{aligned} \tag{2.19}$$

Here the correction function $f\left(|d|\,,p;g\right)$ is given by

$$f\left(|d|\,,p;g\right) = \sum_{k=1}^{\infty} f_k\left(|d|\,,p\right) g^{|d|+k}\,, \tag{2.20}$$

and the similar for $\widetilde{f}\,(|d|,p;g)$. In (2.20), the correction factors $f\left(|x-1|,p;g\right)$ and $f\left(|L-x|,p;g\right)$ tend to vanish as the magnon moves far away from the boundaries. This means that the magnon would not feel the long-range interaction with boundaries if the magnon-wave locates sufficiently far away from both ends, in which case the wavefunction reduces to

$$\psi(x) \approx A(p;g)\,e^{ipx} - \widetilde{A}(-p;g)\,e^{-ipx} \equiv \psi_0(x)\,. \tag{2.21}$$

This is the same ansatz as used in the one-loop analyses of [21, 24, 26, 27]. The reader should remind that the quasi-momentum p may also get a g-correction, and may be expanded as $p(g) = p^{(0)} + gp^{(1)} + g^2p^{(2)} + \ldots$. Another remark is that our PABA (2.19) looks slightly different from the one used in [28], but up to the two-loop level, it results n the same boundary S-matrices as in [28], if applied to the giant graviton system.

Now let us explain the origin of the ansatz (2.19). It is motivated by the PABA for closed spin-chain cases. As introduced in section 1, the PABA technique has been used in various situations, and in the work [28], it was applied to an open spin-chain which describes an open string on a giant graviton. Recall the PABA for the closed spin-chain case, first introduced by Staudacher [33]. The ansatz on a wavefunction for a two-magnon state takes the form

$$\begin{aligned}\psi(x_1,x_2) = \big(1 + F\left(|x_2-x_1|,p_1,p_2;g\right)\big)\,A\left(p_1,p_2;g\right)\ e^{i(p_1x_1+p_2x_2)} + \\ + \big(1 + \widetilde{F}\left(|x_2-x_1|,p_2,p_1;g\right)\big)\,\widetilde{A}\left(p_2,p_1;g\right)\ e^{i(p_2x_1+p_1x_2)}\,,\end{aligned} \tag{2.22}$$

where the correction factor is given by

$$F\left(|d|,p_1,p_2;g\right) = \sum_{k=1}^{\infty} g^{|d|+k} F_k(|d|,p_1,p_2)\,. \tag{2.23}$$

The bulk two-body S-matrix is defined by $S\left(p_2, p_1; g\right) \equiv \widetilde{A}\left(p_2, p_1; g\right) / A\left(p_1, p_2; g\right)$, and admits a g-expansion as follows:

$$S\left(p_1, p_2; g\right)=\sum_{k=0}^{\infty} g^k S^{(k)}\left(p_1, p_2\right) . \tag{2.24}$$

It can be seen that our ansatz (2.19) for a single-magnon wavefunction is constructed so that, at each endpoint, it describes a scattering between a magnon in question at $x_1=x$ with $p_1=p$ and another "magnon" at $x_2=1$ (or $x_2=L$) with $p_2=0$, i.e., the endpoint. Thus we can regard (2.19) as a natural extension of the PABA for a closed spin-chain case, (2.22) with (2.23), to the open spin-chain case, with the two boundaries playing roles of another two rest "magnons".

When we consider the n-loop analysis, by acting the Hamiltonian (2.9) to the Bethe wavefunction with $1+n \leq x \leq L-n$, we have the following eigenvalue equation,

$$\begin{aligned}
\psi(x)\left[\epsilon_1(p) g^1+\epsilon_2(p) g^2+\cdots+\epsilon_n(p) g^n\right]= & \, c_1(g)(2 \psi(x)-\psi(x-1)-\psi(x+1))+ \\
& +c_2(g)(2 \psi(x)-\psi(x-2)-\psi(x+2))+\cdots \\
& \cdots+c_n(g)(2 \psi(x)-\psi(x-n)-\psi(x+n)) .
\end{aligned} \tag{2.25}$$

On the other hand, when we act the Hamiltonian to a state with the magnon at $x=1, \ldots, n$ or $x=L-n+1, \ldots, L$, some of the terms in the right-hand side of (2.25) do not appear. Furthermore the boundary terms in (2.9) give rise to extra terms in the eigenvalue equation. The compatibility conditions that the bulk condition (2.25) persists even at these special points $x=1, \ldots, n$ or $x=L-n+1, \ldots, L$ lead to a set of Bethe equations.

In what follows, we will mainly concentrate on the scattering problem at the left endpoint ($x=1$); the analysis on the right end ($x=L$) can be done in the same manner. The boundary S-matrix is defined as

$$B(p ; g) \equiv \frac{A(-p ; g)}{\widetilde{A}(p ; g)}, \tag{2.26}$$

and will be expanded as

$$B(p ; g)=\sum_{k=0}^n B^{(k)}(p)\, g^k+\mathcal{O}\left(g^{n+1}\right) . \tag{2.27}$$

As mentioned just before, special care has to be payed to $x=1,2, \ldots, n$ cases, where the boundary terms will affect the eigenvalue equations. The compatibility conditions lead to a set of constraints among coefficients $\{f_k\}$ and $\{\widetilde{f}_k\}$ $(k=1, \ldots, n-1)$. Using those equations, B can be expressed in terms of the quasi-momentum p in the ansatz (2.19) and unfixed coefficients $C_{x, x+s}^{(k)}$ in the boundary Hamiltonian (2.7).

Let us see how the PABA (2.19) works in the concrete case of $n=2$, i.e., the two-loop analysis. Compatibility conditions are calculated as, for $x=1$,

$$\begin{aligned}
2 \psi_0(0)-\left(2-C_{1,1}^{(1)}\right) \psi_0(1)=g\Big[& \left(-2+\epsilon_1(p)-C_{1,1}^{(1)}\right) e^{i p} f_1(p) B(-p ; g)+ \\
& +\left(2-\epsilon_1(p)+C_{1,1}^{(1)}\right) e^{-i p} \widetilde{f}_1(-p)- \\
& -2 \psi_0(-1)+8 \psi_0(0)-\left(6+C_{1,1}^{(2)}\right) \psi_0(1)-C_{1,2}^{(2)} \psi_0(2)\Big],
\end{aligned} \tag{2.28}$$

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and for $x = 2$,

$$2B(-p;g)\, e^{ip} f_1(p) - 2e^{-ip} \widetilde{f}_1(-p) - 2\psi_0(0) - C^{(2)}_{2,1}\psi_0(1) - \big(-2 + C^{(2)}_{2,2}\big)\psi_0(2) = 0\,. \quad (2.29)$$

We have similar conditions for $x = L-1$ and $x = L$, and the compatibility conditions at $x = 3, 4, \ldots, L-2$ become trivial. We can solve (2.29) for $\widetilde{f}_1$, then plugging it into (2.28) along with (2.21) (the definition of ψ_0), we are left with one p-dependent condition written in term of B and C's. Solving it for B and expanding it as (2.27), we can determine the coefficients of the series $B^{(0)}$ and $B^{(1)}$ perturbatively. Hence the resulting $B^{(k)}$ are functions of a quasi-momentum p and the undetermined coefficients C's in the Hamiltonian. We should note that $p(g)$ will further allow a series expansion in g, as we will see later.

Going to the three-loop order ($n = 3$), we face a curious situation. The compatibility condition at $x = 3$ together with (2.29) leads to

$$\Big(-2C^{(2)}_{2,1} + C^{(3)}_{3,1}\Big)\, \psi_0(1) + \Big(4 + C^{(3)}_{3,2} - 2C^{(2)}_{2,2}\Big)\, \psi_0(2) + \Big(-4 + C^{(3)}_{3,3}\Big)\, \psi_0(3) = 0\,, \quad (2.30)$$

which can be at all satisfied at least for $C^{(3)}_{3,3} = 4$; otherwise the value of the quasi-momentum would be forced to take special values, which would make no sense. As we will see in the next section, an open spin-chain related to a giant graviton case is indeed shown to be endowed with $C^{(3)}_{3,3} = 4$, so there is no problem about the validity of our Bethe ansatz at least up to three-loop level. As for the dCFT case, on the other hand, we have no existing perturbative calculation beyond the one-loop order, so we have no positive evidence for the consistency. But assuming the integrability of the model, in the dCFT case also, one may expect that (2.29) is again satisfied trivially with a nice set of coefficients.

In the rest part of the paper, we will apply the PABA introduced in this section to two concrete examples.

3. PABA for Open Spin-Chains in Giant Graviton System

Let us first consider an open spin-chain description of an open string on a giant graviton. There are numerous number of papers concerning the system; for some of them, see [40–45]. A giant graviton is a rotating spherical D3-brane which is wrapping an $\mathrm{S}^3 \subset \mathrm{S}^5$ of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, and a giant graviton maximally enhanced in S^5 is especially called a maximal giant graviton. This is a BPS object described by a determinant-like operator made up of a single complex scalar field in $\mathcal{N} = 4$ SYM, say Z. The determinant-like operator is called a giant graviton operator. For the maximal giant graviton case we can write it as $\epsilon^{j_1 \ldots j_N}_{i_1 \ldots i_N} Z^{i_1}_{j_1} \ldots Z^{i_N}_{j_N}$, where the symbol $\epsilon^{j_1 \ldots j_N}_{i_1 \ldots i_N}$ is understood as a product of antisymmetric tensors $\epsilon^{j_1 \ldots j_N}\epsilon_{i_1 \ldots i_N}$. Attaching an "open string" with length-$(L+2)$ to the maximal giant gives an operator of the form

$$\epsilon^{j_1 \ldots j_N}_{i_1 \ldots i_N} Z^{i_1}_{j_1} \ldots Z^{i_{N-1}}_{j_{N-1}} \, (M_0 M_1 \ldots M_L M_{L+1})^{i_N}_{j_N}\ , \quad (3.1)$$

with the matrix product $M_0 \ldots M_{L+1}$ describing the open string. We consider the SU(2) sector, where each M_l can be either Z or Y that are two of the three complex holomorphic scalars in $\mathcal{N} = 4$ SYM.

One of the important constraints in our analysis is that we have a boundary condition such that neither M_0 nor M_{L+1} can be Z, since otherwise the operator (3.1) factorizes into a maximal giant and a closed string, which drops in the large-N analysis [24, 45]. We can define the Bethe reference state, or the ground state, as the highest weight state under the unbroken SO(4). At this time we can take it as

$$\epsilon^{j_1 \dots j_N}_{i_1 \dots i_N} Z^{i_1}_{j_1} \dots Z^{i_{N-1}}_{j_{N-1}} \left(Y^{L+2} \right)^{i_N}_{j_N} . \tag{3.2}$$

Let us now consider a single-magnon excitation on the chain. We concentrate on the operators of the type

$$|x\rangle \equiv Y \big[\underbrace{Y \dots Y \overset{\overset{x}{\downarrow}}{Z} Y \dots Y}_{L} \big] Y . \tag{3.3}$$

with one Y at $l = x$ in (3.1) replaced with Z. As discussed before, the boundary condition indicates $x \neq 0,\ L+1$, which in turn implies vanishing of the Bethe wavefunction of Z in the vicinities of $x = 0,\ L+1$.

To find the boundary coefficients $C^{(k)}_{x,x+s}$, let us split the first term in (2.9) into three pieces:

$$\sum_{l=0}^{L} c_1(g) Q_{l,l+1} = c_1(g) \left[Q_{0,1} + \sum_{l=1}^{L-1} Q_{l,l+1} + Q_{L,L+1} \right] . \tag{3.4}$$

As done above, we have to treat $Q_{0,1}$ and $Q_{L,L+1}$ with special care, since they are the only possible terms in the sum relevant to the boundary conditions. We should be careful in acting these two pieces to the length-$(L+2)$ word $\boldsymbol{w} = M_0 M_1 \dots M_L M_{L+1}$ with $M_0 = Y$ and $M_{L+1} = Y$ because of the boundary conditions. For example, in the case that $Q_{0,1}$ acts on $\boldsymbol{w}$,

$$Q_{0,1} \boldsymbol{w} = (I - P)_{0,1} \left[Y M_1 M_2 \dots M_{L+1} \right] = \left(Y M_1 - M_1 Y \right) M_2 \dots M_{L+1} , \tag{3.5}$$

but the state with $M_0 = Z$ is suppressed in the large N limit, so the expression above reduces to $YZM_2 \dots M_{L+1}$ if $M_1 = Z$. Or else, if $M_1 = Y$, we see that $Q_{0,1}\boldsymbol{w}$ vanishes. The similar is true for the action of $Q_{L,L+1}$ to $\boldsymbol{w}$. Thus we can write

$$\left[\sum_{l=0}^{L} c_1(g) Q_{l,l+1} \right] \boldsymbol{w} = c_1(g) \sum_{l=1}^{L-1} Q_{l,l+1} \boldsymbol{w} + c_1(g) \left(q^Z_1 + q^Z_L \right) \boldsymbol{w} , \tag{3.6}$$

where we have introduced a projection operator q^X_k which gives one when the field on the k-th site is X, and otherwise zero. In the same way, we can express the action of the second term in (2.9) on $\boldsymbol{w}$ as

$$\left[\sum_{l=0}^{L-1} c_2(g) Q_{l,l+2} \right] \boldsymbol{w} = c_2(g) \sum_{l=1}^{L-2} Q_{l,l+2} \boldsymbol{w} + c_2(g) \left(q^Z_2 + q^Z_{L-1} \right) \boldsymbol{w} . \tag{3.7}$$

In these ways, we can write down the action of the Hamiltonian to general state $\boldsymbol{w}$ composed of two holomorphic scalars, whose first and the last sites cannot be a magnon Z, in terms of operators $Q_{l,l+k}$ and q_k^Z. This is the case even for a general M-magnon problem; we have only to add as many boundary terms of the form q_k^Z as the number of magnons, only two of which can be relevant to the boundary interaction at this two-loop order.

We should keep in mind that this feature we have seen so far is limited to the analysis up to two-loop level. Three-loop interaction introduces QQ-term as in (2.2), which can change the array of magnons between in- and out-states. For example, in the $M = 2$ case, when we begin with the in-state with magnons at $l = 1$ and $l = 2$, which we denote as $|1, 2\rangle$, the operation of $Q_{0,1}Q_{2,3}$ mixes $|1, 2\rangle$ and $|1, 3\rangle$ in the out-state. Symbolically, $QQ\,|x_1, x_2\rangle$ is in general not proportional to the original word $|x_1, x_2\rangle$ due to the boundary interactions. In such cases, the boundary Hamiltonian cannot be given mealy by projection operators.

Nevertheless, even beyond two-loop analysis, as long as we consider a single-magnon problem, the QQ-term never affect the array in the word, and so we can write, for example, the action of the third term in (2.9) as

$$\left[\sum_{l=0}^{L-2} c_2(g) Q_{l,l+3}\right] \boldsymbol{w} = c_3(g) \sum_{l=1}^{L-3} Q_{l,l+3}\boldsymbol{w} + c_3(g)\left(q_3^Z + q_{L-2}^Z\right)\boldsymbol{w}\,. \tag{3.8}$$

Let us return to the $M = 1$ case and concentrate on the two-loop analysis. In this case, as mentioned before, acting the boundary Hamiltonian to a single-magnon state $|x\rangle$ gives only a state proportional to $|x\rangle$, i.e., we have

$$\text{coeff. of } g^1\text{:}\quad C_{1,1}^{(1)} = 2\,; \tag{3.9}$$

$$\text{coeff. of } g^2\text{:}\quad C_{1,1}^{(2)} = -8\,,\quad C_{1,2}^{(2)} = 0\,,\quad C_{2,1}^{(2)} = 0\,,\quad C_{2,2}^{(2)} = 2\,; \tag{3.10}$$

$$\begin{aligned}\text{coeff. of } g^3\text{:}\quad & C_{1,1}^{(3)} = 60\,,\quad C_{1,2}^{(3)} = 0\,,\quad C_{1,3}^{(3)} = 0\,,\quad C_{2,1}^{(3)} = 0\,,\quad C_{2,2}^{(3)} = -24\,,\\ & C_{2,3}^{(3)} = 0\,,\quad C_{3,1}^{(3)} = 0\,,\quad C_{3,2}^{(3)} = 0\,,\quad C_{3,3}^{(3)} = 4\,.\end{aligned} \tag{3.11}$$

Note that the compatibility condition (2.30) is satisfied within the coefficients (3.10) and (3.11). Following the procedure explained in section 1, the boundary S-matrix $B(p; g)$ defined at $x = 0$ can be obtained perturbatively.

The first series coefficient turns out to be

$$B^{(0)}(p) = \frac{2 - e^{-ip}\big(2 - C_{1,1}^{(1)}\big)}{2 - e^{ip}\big(2 - C_{1,1}^{(1)}\big)}\,, \tag{3.12}$$

and thus we can see (3.9) indeed ensures $B^{(0)} = 1$. The compatibility condition at $x = L+1$ together with (3.12) gives,

$$1 = e^{2ip(L+1)} \cdot \frac{2 - e^{-ip}\big(2 - C_{1,1}^{(1)}\big)}{2 - e^{ip}\big(2 - C_{1,1}^{(1)}\big)} \cdot \frac{2 - e^{-ip}\big(2 - C_{L,L}^{(1)}\big)}{2 - e^{ip}\big(2 - C_{L,L}^{(1)}\big)}\,, \tag{3.13}$$

which leads to, at the one-loop order, the following momentum quantization condition

$$p = \frac{n\pi}{L+1} + \mathcal{O}(g^1)\,, \tag{3.14}$$

where we have used the fact $C^{(1)}_{1,1} = 2$ and $C^{(1)}_{L,L} = 2$. Here n is an integer which labels a momentum index. Thus the Bethe wavefunction is given by

$$|\Psi(p)\rangle = a_n \sum_{1\le x\le L} \sin\left[\frac{\pi n}{L+1}\, x\right] |x\rangle \,, \tag{3.15}$$

with an n-dependent normalization factor a_n. We can see this wavefunction indeed vanishes at the endpoints $x = 0$ and $x = L+1$, and it gives us an intuitive picture of an open string attaching to a giant graviton, which is subject to a Dirichlet boundary condition. These one-loop analyses were first done by Berenstein and Vazquez [24]. Does this matching of boundary conditions still hold at higher order in perturbation theory? Or, put it differently, can we find the dictionary to relate certain directions in the string target spacetime to certain fields in the SYM theory, even at the higher-loop level? This is our initial motivation to start the PABA analyses of open spin-chains resulting from the SYM theory.

As shown below, when we consider higher-loop corrections to the boundary S-matrices, the boundary S-matrices defined at $x = 0$, $L+1$ indeed cease to be one due to the higher-loop contributions. In view of AdS/CFT correspondence, it would be then reasonable to regard somewhere near $x = 0$, $L+1$, at which the wavefunction vanishes, as a perturbatively corrected definition of the endpoints. This redefinition of the endpoints provides a nice picture even at higher-loop level comparable to the behavior of an open string attaching to a giant graviton, which obeys exactly the Dirichlet boundary condition in the strong-coupling region. Or we may be able to interpret the perturbative g-correction as implications of an interaction between an open string and a brane. It would be interesting to confirm this from the direct analysis of the Dirac-Born-Infeld action.

We argue that the locations of the endpoints should be corrected by a pure phase factor of the form $e^{i\theta(p;g)}$ perturbatively in g, and assume the following series expansion:

$$\theta\,(p; g) = \sum_{k=0}^{\infty} \theta_k(p) g^k \,. \tag{3.16}$$

Then, by defining

$$B_{\text{left}}(p; g) \equiv e^{i\theta(p;g)} B_0(p; g)\,, \qquad B_{\text{right}}(p; g) \equiv e^{-i\theta(p;g)} B_{L+1}(p; g)\,, \tag{3.17}$$

the momentum condition can be expressed as

$$1 = e^{2i[p(L+1)-\theta(p;g)]} B_{\text{left}}(p; g) B_{\text{right}}(-p; g)\,. \tag{3.18}$$

Demanding both $B_{\text{left}}(p; g)$ and $B_{\text{right}}(p; g)$ to be one, the series coefficients of the correction factor are determined as

$$\theta_0(p) = 0\,, \quad \theta_1(p) = -4 \sin p^{(0)}\,, \tag{3.19}$$

where we have expanded the quasi-momentum as $p(g) = p^{(0)} + g p^{(1)} + g^2 p^{(2)} + \dots$. Solving $1 = \exp\{2i\,[p\,(L+1) - \theta(p; g)]\}$ perturbatively, we get

$$p(g) = \frac{n\pi}{L+1} + \frac{4g}{L+1} \sin\left(\frac{n\pi}{L+1}\right) + \mathcal{O}(g^2)\,. \tag{3.20}$$

This is the same result as obtained in [28] earlier. We see that, in the two-loop analysis, where to identify with an endpoint has been corrected from $x = 0$ of the one-loop analysis to $x = 0 - 2g \sin p^{(0)}/p^{(0)}$ with $p^{(0)} = n\pi/(L+1)$. Similarly, the right endpoint should be identified with $x = L+1+2g\sin p^{(0)}/p^{(0)}$. The boundary S-matrices $B_{\text{left/right}}(p)$ are thus defined at these perturbatively corrected endpoints to give one.

Finally we can calculate the corrected energy of the single-magnon state, or the anomalous dimension of the SYM operator with one impurity, as

$$\begin{aligned}E(n;\lambda,L) = \frac{n^2\lambda}{8L^2}\left(1-\frac{2}{L}+\frac{36-n^2\pi^2}{12L^2}+\dots\right) - \frac{n^4\lambda^2}{128L^4}\left(1-\frac{4}{L}+\frac{60-n^2\pi^2}{6L^2}+\dots\right) - \\ - \frac{n^2\lambda^2}{16\pi^2L^3}\left(1-\frac{3}{L}+\frac{18-n^2\pi^2}{3L^2}+\dots\right) + \mathcal{O}(\lambda^3)\,,\end{aligned} \tag{3.21}$$

where we have used $\lambda = 16\pi^2 g$. The first line of (3.21) results from $\epsilon_1 + \epsilon_2\big|_{p=p^{(0)}}$, and the second from the $\mathcal{O}(g)$-correction term in (3.20). This is somewhat an unexpected result, since the latter contribution diverges in the BMN limit, i.e., $L \to \infty$ with λ/L^2 kept fixed.

One may suspect that the apparent breakdown of the BMN scaling results from a wrong counting of the N-dependence of the normalization factor for the states $\boldsymbol{w}$ when studying the region $\sqrt{N} \sim L$. As discussed before, actually we neglected all the states whose first or last site is occupied by the field Z, which is the field forming the giant graviton. According to the work [28] that studied the N-dependence of the normalization carefully by using the Hamiltonian of [36], however, those states with Z at first or last site should only contribute at the subleading order in $1/N$, hence $1/L^2$, therefore, they should be irrelevant to the breakdown.

After this paper, it is argued in [46] that the apparent breakdown of the BMN scaling we have seen implies a breakdown of applicability of a Bethe Ansatz, and that the integrability itself as well as the BMN scaling do exist. It would be interesting to follow the direction of [46] to further examine the problem.

4. PABA for Open Spin-Chains in Defect Conformal Field Theory

Next we turn to a system of an open spin-chain that appears in a defect field theory (dCFT). We are interested in a D3-D5 system, with a stack of N coincident D3-branes with one D5-brane probe, the setup first considered by Karch and Randall [47]. The action of the dCFT dual to the D3-D5 system was constructed by DeWolfe, Freedman and Ooguri [39] and the superconformality in the non-abelian case was studied in [48]. The original $\mathcal{N} = 4$ SYM theory contains six real scalars X^i and adjoint Majorana spinors λ^α which transform **6** and **4** of the SO(6) R-symmetry, and there is also a gauge field A_μ. In the D3-D5 system, the D3-branes fill the 0126-directions, while the probe D5-brane spans 012345 (see the table below).

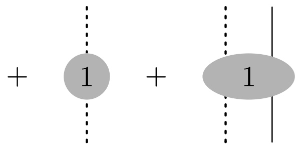
	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	○	○	○	×	×	×	○	×	×	×
D5	○	○	○	○	○	○	×	×	×	×

There is a three-dimensional defect in 012-directions, which introduces a three-dimensional $\mathcal{N} = 4$ hypermultiplet in additional to the bulk four-dimensional hypermultiplet. This defect preserves $\mathrm{SO}(3,2)$ subgroup of the four-dimensional conformal group $\mathrm{SO}(4,2)$, and breaks the R-symmetry $\mathrm{SO}(6)$ to $\mathrm{SO}(3)_{\mathrm{H}} \times \mathrm{SO}(3)_{\mathrm{V}}$. The bulk fields are decomposed into a three-dimensional vector multiplet $\{A_k, P_+\lambda^\alpha, X_{\mathrm{V}}^A, D_3 X_{\mathrm{H}}^I\}$ and a three-dimensional hypermultiplet $\{A_3, P_-\lambda^\alpha, X_{\mathrm{H}}^I, D_3 X_{\mathrm{V}}^A\}$ with $k = 0, 1, 2$, $A = 1, 2, 3$ and $I = 4, 5, 6$. The X_{H} and X_{V} are real scalar fields which are transformed as $(\mathbf{3}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{3})$ of $\mathrm{SO}(3)_{\mathrm{H}} \times \mathrm{SO}(3)_{\mathrm{V}}$. The four-dimensional Majorana spinor is split into three-dimensional Majorana spinors $P_\pm\lambda^\alpha$, where $P_\pm$ are projection operators, and $\lambda^\alpha \mapsto \lambda_{am}$ are arranged to transform as $(\mathbf{2}, \mathbf{2})$.

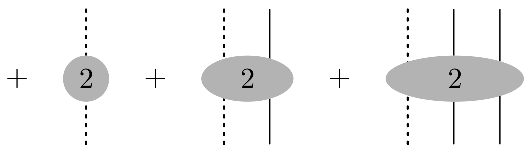
We are interested in the SU(2) sector, where composite operators are made up of two holomorphic scalars $Z \equiv X_{\mathrm{H}}^1 + iX_{\mathrm{H}}^2$ and $W \equiv X_{\mathrm{V}}^4 + iX_{\mathrm{V}}^5$, and the latter plays the role of a magnon. In addition to these fields, the three-dimensional defect fields $\overline{q}_1$ and q_2 are needed to form a Bethe state of our concern (see (4.5)).

First let us examine the structure of the dilatation operator, or the spin-chain Hamiltonian, for this sector. As discussed in section 1, it is given by

$$H = \sum_{l=1}^{L-1} c_1(g) Q_{l,l+1} + \sum_{l=1}^{L-2} c_2(g) Q_{l,l+2} + \sum_{l=1}^{L-3} c_3(g) Q_{l,l+3} \tag{4.1}$$



(4.2)



(4.3)

$+ \cdots$,

where the second and the third lines of the above Hamiltonian represent the boundary interactions at the one- and the two-loop order, respectively. We omit to write the third-order diagram. The solid and the dotted lines represent a four-dimensional bulk and a three-dimensional defect fields, respectively. The one-loop contribution (4.2) was calculated in [21, 27], and at this level, the location of the magnon W does not change, which can be easily seen from the Feynman diagrams. Thus the resulting expression for the boundary Hamiltonian can be expressed in terms of projection operators, and is given by

$$(4.2) = 4g \left(q_1^W + q_L^W \right) , \tag{4.4}$$

i.e., we have $C_{1,1}^{(1)} = 4$ in (2.8).

From the two-loop order, however, the location of W can in general change between an in-state and an out-state. The third diagram in (4.3) contains such diagrams, which have a $qqXX$- and a $4X$-vertex (X is a four-dimensional scalar Z or W, and q is a three-dimensional scalar). Currently we do not know the values of $C_{1,1}^{(2)}$, $C_{1,2}^{(2)}$, $C_{2,1}^{(2)}$ and $C_{2,2}^{(2)}$, and to determine them, it is in principle necessary to perform direct two-loop order perturbative calculations for the boundary interaction. But it seems a hard task, so for now, we decide to leave them as free parameters.

For our purpose to study the boundary S-matrix and the energy spectrum via the PABA, we shall define a Bethe reference state as $\overline{q}_1 Z^L q_2$, and set the following single-magnon state in (2.19):

$$|x\rangle \equiv \overline{q}_1 \underbrace{Z \dots Z \overset{\overset{x}{\downarrow}}{W} Z \dots Z}_{L} q_2 \,. \tag{4.5}$$

In contrast to the case of an open spin-chain on a giant graviton, for the defect operators (4.5), we define the boundary S-matrix at $x = 1/2$ and $x = L + (1/2)$, reflecting the fact that each of defect fields $\overline{q}_1$ and q_2 has a bare dimension $1/2$. Namely, the magnon wave sees an effective length-$(L+1)$ chain. The first series coefficient of the boundary S-matrix defined at $x = 1/2$ is calculated as

$$B^{(0)} = \frac{2e^{ip} - \left(2 - C^{(1)}_{1,1}\right)}{2 - e^{ip}\left(2 - C^{(1)}_{1,1}\right)} \,, \tag{4.6}$$

but since we already know that $C^{(1)}_{1,1} = 4$, the right-hand side of the above equation vanishes, i.e., $B^{(0)} = 1$. This is true for the right endpoint, too. The momentum condition becomes

$$1 = e^{2ipL} \cdot \frac{2e^{ip} - \left(2 - C^{(1)}_{1,1}\right)}{2 - e^{ip}\left(2 - C^{(1)}_{1,1}\right)} \cdot \frac{2e^{ip} - \left(2 - C^{(1)}_{L,L}\right)}{2 - e^{ip}\left(2 - C^{(1)}_{L,L}\right)} \,, \tag{4.7}$$

which leads to, at the one-loop order,

$$p = \frac{n\pi}{L} + \mathcal{O}(g^1) \tag{4.8}$$

with the mode index n. The Bethe wavefunction is given by

$$|\Psi(p)\rangle = a_n \sum_{1 \le x \le L} \sin\left[\frac{\pi n}{L}\left(x - \frac{1}{2}\right)\right] |x\rangle \,, \tag{4.9}$$

which indeed vanishes at the endpoints $x = 1/2$, $L + (1/2)$, as expected from the behavior of the dual open-string attaching to a brane extended in the Z-direction [21, 27].

Let us continue the study to higher loops. The analysis essentially resembles the case of an open string on a giant graviton in the previous section. The momentum condition becomes

$$1 = e^{2i[pL - \theta(p;g)]} B_{\text{left}}(p;g) B_{\text{right}}(-p;g) \tag{4.10}$$

with the similar definitions of $B_{\text{left/right}}(p;g)$ as in (3.17). Demanding, again, $B_{\text{left}}(p;g) = 1$ and $B_{\text{right}}(p;g) = 1$, we get

$$\theta_0(p) = 0 \,, \tag{4.11}$$

$$\begin{aligned}\theta_1(p) = -\frac{1}{2}\tan\left(\frac{p^{(0)}}{2}\right)\Big[4\left(1 - 4\cos(p^{(0)}) - \cos(2p^{(0)})\right) + C^{(2)}_{1,1} + \\ + \left(1 + 2\cos p^{(0)}\right)\left(C^{(2)}_{1,2} + C^{(2)}_{2,1}\right) + \left(1 + 2\cos p^{(0)}\right)^2 C^{(2)}_{2,2}\Big] \,.\end{aligned} \tag{4.12}$$

Solving $1 = \exp\{2i[pL - \theta(p;g)]\}$ perturbatively, we get

$$p(g) = \frac{n\pi}{L} + \frac{g}{2L}\tan\left(\frac{n\pi}{2L}\right)\left[4 + C^{(2)}_{1,1} + C^{(2)}_{1,2} + C^{(2)}_{2,1} + 3C^{(2)}_{2,2} + 2\left(-2 + C^{(2)}_{2,2}\right)\cos\left(\frac{2n\pi}{L}\right) + \right.$$
$$\left. + 2\left(-8 + C^{(2)}_{1,2} + C^{(2)}_{2,1} + 2C^{(2)}_{2,2}\right)\cos\left(\frac{n\pi}{L}\right)\right] + \mathcal{O}(g^2)\,. \tag{4.13}$$

Recall that we have the condition $C^{(2)}_{1,2} = C^{(2)}_{2,1}$ because of the Hermiticity of the Hamiltonian. Then the above expression tells us that, the quasi-momentum does not depend on the gauge coupling if and only if the coefficients are given by, in addition to $C^{(1)}_{1,1} = 4$,

$$C^{(2)}_{1,1} = -14\,, \quad C^{(2)}_{1,2} = C^{(2)}_{2,1} = 2 \quad \text{and} \quad C^{(2)}_{2,2} = 2\,; \tag{4.14}$$

otherwise, they generally receive g-corrections as in the giant graviton case. This set of coefficients leads to vanishing $\theta_1(p)$ in (4.12), too. Note also, the last condition in (4.14) is compatible with the condition for our PABA to be valid even at the three-loop order, which we discussed at the end of section 1. Of course, up to now, there is no reason one can believe the perturbative calculation should result in (4.14), but these values are very plausible from the viewpoint of the integrability. As we will see soon, the condition (4.14) also turns out a sufficient condition for the system to have a smooth BMN limit. These points incline us to believe (4.14) really predicts the result of direct perturbative computation. It is an interesting task to carry out the two-loop computation directly in order to check our argument.

Finally, the single-impurity state energy is calculated as

$$\begin{aligned} E(n;\lambda,L) = {} & \frac{n^2\lambda}{8L^2}\left(1 - \frac{n^2\pi^2}{12L^2} + \frac{n^4\pi^4}{360L^4} + \ldots\right) \\ & - \frac{n^4\lambda^2}{128L^4}\left(1 - \frac{n^2\pi^2}{6L^2} + \ldots\right) + \\ & + \frac{n^2\lambda^2}{256\pi^2L^3}\left(-16 + C^{(2)}_{1,1} + 3C^{(2)}_{1,2} + 3C^{(2)}_{2,1} + 9C^{(2)}_{2,2}\right) - \\ & - \frac{n^4\lambda^2}{3072L^5}\left(-208 + C^{(2)}_{1,1} + 15C^{(2)}_{1,2} + 15C^{(2)}_{2,1} + 81C^{(2)}_{2,2}\right) + \mathcal{O}(\lambda^3)\,. \end{aligned} \tag{4.15}$$

In (4.15), the first line stands for the one-loop energy $\epsilon_1(p^{(0)})$, and the following three lines for the two-loop energy; the first of which resulting from $\epsilon_2(p^{(0)})$, and the rest depending on the first corrected piece in the momentum, $p^{(1)}$. For (4.15) to obey the BMN scaling, we see the third line has to vanish, since it diverges in the BMN limit. Interestingly, this condition is satisfied within the choice of the special coefficients (4.15). Of course, this choice of coefficients then leads to the vanishing forth line of (4.15), making the $\mathcal{O}(g^2)$-energy totally $p^{(1)}$-independent.

Finite size-corrections for an open spin-chain in the dCFT was also calculated by McLoughlin and Swanson [25], but the contributions obtained there resulted only from interactions between more than one magnons, which is clearly different source from ours

(4.15). In fact, the authors of [25] treated boundary S-matrices trivially, which would amount to set (4.14) from the beginning. If one finds, in the future calculations of the dCFT, that (4.14) is not really the case, then one would have to take account of the finite-size corrections of (4.15) in addition to the contribution calculated in [25].

5. Summary and outlook

In this paper, we have proposed a perturbative asymptotic Bethe ansatz for open spin-chain systems that have been derived from particular field theory structures and symmetries. By considering single-magnon problems for the open spin-chains, we have analyzed the boundary S-matrices using the PABA method. As concrete examples, we have considered two models; one is related to a giant graviton operator (3.1) and the other to a defect operator (4.5).

In the giant graviton case, we have evaluated the numerical coefficients in the boundary Hamiltonian explicitly, which reflect the contributions of higher-loop diagrams. This result has enabled us to directly calculate higher-loop corrections to the boundary S-matrix B as well as in the quasi-momentum p, with our PABA ansatz (2.19) and (2.20). Then it has been shown that the higher-loop contribution introduced g-dependence in both B and p, just as was shown in [28]. As the result, the single-magnon energy acquires a divergent piece in the BMN limit, the physical interpretation of which remains to be clarified.

On the other hand, in the dCFT case, there has been no known results concerning the values of the boundary coefficients. Though these should be determined by direct perturbative computation, we have not tried to carry out the perturbative calculation in this paper. Instead we have performed the PABA analysis by leaving them arbitrary constants. We have found that B's and g, again, depend on g in general. But, in contrast to the giant graviton case, it is possible to take the coefficients so that B's and g become independent of g. If the future direct perturbative calculation really reproduces the special set of coefficients (4.14), the energy of the single-magnon state turns out to have a smooth BMN limit, but otherwise it diverges in this limit. One should, however, bare in mind that the conclusion made above can only be right when the compatibility conditions are fulfilled to make the Bethe ansatz valid. When the coefficients turned out to take different values from (4.14) and did not satisfy the compatibility conditions, the PABA we employed in section 3 is not valid and it cannot be used to study the energy spectrum of the giant graviton system.

We have also shown in section 2 that an open spin-chain system can have a perturbative integrability beyond the two-loop order, if and only if the conditions $2C_{2,1}^{(2)} - C_{3,1}^{(3)} = 0$, $4 + C_{3,2}^{(3)} - 2C_{2,2}^{(2)} = 0$ and $C_{3,3}^{(3)} = 4$ are satisfied at the same time, see (2.30). The coefficients for the giant graviton case, (3.10) and (3.11), indeed respect these relations. Hence, as long as a single-magnon problem is concerned, there is no apparent breakdown of integrability at least up to the three-loop level, as in the closed spin-chain case. (Here the terminology "integrable" means that the system can be solved via the Bethe ansatz). However, see [28], which implied the non-integrability of the giant-graviton system by studying a two-magnon state at the two-loop order; the meaning of the "integrability" in [28] slightly differs from our present paper, though.

It would be very interesting to study quantum corrections on the string side, by directly analyzing the Dirac-Born-Infeld action. In [31], a dictionary between gauge-invariant BMN operators in the $\mathcal{N}=4$ SYM and the dual open string states in the pp-wave background was proposed by Balasubramanian, Huang, Levi and Naqvi. There the Bethe reference state corresponds to the light-cone vacuum, and the excited state created by a creation operator $a_n^{\dagger\, Z}$ in Z-direction with the mode number n corresponds to a superposition of single-impurity (Z) states with appropriate sine phase-factor insertion. Explicitly, the dictionary reads (dropping the normalization factors)

$$|-1,0\rangle_{\rm l.c.} \quad\longleftrightarrow\quad \epsilon^{j_1\ldots j_N}_{i_1\ldots i_N} Z^{i_1}_{j_1}\ldots Z^{i_{N-1}}_{j_{N-1}} \left(Y^{L+2}\right)^{\,i_N}_{\,j_N}\,, \tag{5.1}$$

$$a_n^{\dagger\, Z}\,|-1,0\rangle_{\rm l.c.} \quad\longleftrightarrow\quad \epsilon^{j_1\ldots j_N}_{i_1\ldots i_N} Z^{i_1}_{j_1}\ldots Z^{i_{N-1}}_{j_{N-1}} \left(\textstyle\sum_{l=0}^{L+1}\sin\left(\frac{n\pi l}{L+1}\right)\, Y^l Z Y^{L-l+1}\right)^{i_N}_{j_N}. \tag{5.2}$$

There exists a known dictionary in the dCFT case, too. In [49], Lee and Park constructed a dictionary between gauge invariant BMN operators in the dCFT and the dual open string states in the pp-wave background, which reads:

$$\left|0;p^+\right\rangle_{\rm l.c.} \quad\longleftrightarrow\quad \overline{q}_1 Z^L q_2\,, \tag{5.3}$$

$$a_n^{\dagger\, W}\left|0;p^+\right\rangle_{\rm l.c.} \quad\longleftrightarrow\quad \textstyle\sum_{l=0}^{L}\sin\left(\frac{n\pi l}{L}\right)\,\overline{q}_1 Z^l W Z^{L-l} q_2\,. \tag{5.4}$$

Our PABA analysis has given g-corrections to the sine factors in the right-hand sides of (5.2) and (5.4), that is, $p^{(0)}=n\pi/(L+1)$ in (5.2) and $p^{(0)}=n\pi/L$ in (5.4) are corrected to $p^{(0)}+gp^{(1)}+\mathcal{O}(g^2)$ as in (3.20) and (4.13), respectively. Then it would be interesting to study how the left-hand sides of the dictionary will receive corrections in the 't Hooft coupling λ and the inverse tension α'.

We hope the PABA technique for open spin-chains introduced in this paper could be applied to other examples as well, so that it would enable us to capture a deeper nature of open spin-chains, and also of open strings on a brane, in light of the AdS/CFT correspondence.

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